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SIMILARITY OF HEAT- AND MASS-TRANSFER PROCESSES IN DRYING

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The possibility of the existence of a hydrodynamically invariant relation between individual parameters of the process of convective drying is studied. The presence of a relation of this kind is a consequence of the similarity of processes that is typical of regular modes of heat and mass transfer.

Among the variety of interrelated processes of heat and mass transfer, of special interest are the cases where a functional relation independent of the conditions of the process can be established a priori between individual parameters. For example, Fig. 1 presents the dependence of the change in the temperature of a wet body (peat slab) on its humidity in convective drying. The data are taken from [1]. We note that within the range of air velocities from 0 to 12 m/sec the considered relation is a solid curve within the limits of the experimental error, i.e., it is virtually invariant to the hydrodynamics of the process. We can give any number of such examples.

Unfortunately, it is impossible to explain the presence of these facts on the basis of the known model representations, which makes their use limited; nevertheless, the practical value of relations of this kind is of no doubt. Therefore, we will try to reveal the reasons and find the conditions for the existence of an invariant local relation between the temperature and the humidity of a dried body. We assume that there exists a functional dependence that is unique for all points of the body (UFD) of the type $t_s = \varphi(u)$, which does not depend on the intensity of the flows. The invariant relation in nongradient drying is an example [2].

The presence of a UFD can be treated as follows. The body surface serves as a generator of interrelated fields of temperature and humidity. Any combination of t_s and u appears first on the surface, and then it passes into the depth of the body with its previous values. Thus, to retain the UFD in displacement of the fields, they must be tightly connected with each other, i.e., the velocities of displacement of the fields of temperature and humidity must be the same: $v_t = v_u$.

The flows of moisture inside the body (the total flow j and that of vapor j_1) are described by the known equations [1]

$$j = -a_m (\nabla u + \delta \nabla t_s)\,, \quad j_1 = -a_{m1} (\nabla u + \delta_1 \nabla t_s)\,. \tag{1}$$

In the presence of the UFD, we have (the subscript 1 refers to the vapor phase): $t_s = \varphi(u)$; $\nabla t_s = \varphi' \nabla u$, where $\varphi' = dt_s/du$. Then

$$j = -a_m (1 + \delta\varphi') \nabla u\,, \quad j_1 = -a_{m1} (1 + \delta_1\varphi') \nabla u\,. \tag{2}$$

Since δ_1, δ, and φ' are the parameters of state, i.e., they are completely determined by the values of t_s and u at the considered point, they can be united into generalized (effective) coefficients of transfer

$$a_m^{\text{eff}} = a_m (1 + \delta\varphi')\,, \quad a_{m1}^{\text{eff}} = a_{m1} (1 + \delta_1\varphi')\,. \tag{3}$$

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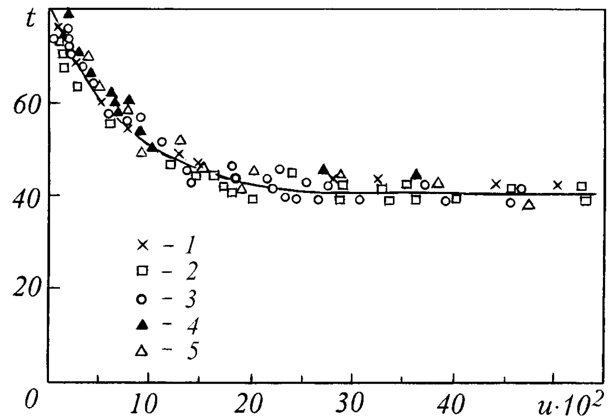


Fig. 1. Mean temperature as a function of moisture content in drying of a peat slab for different velocities of motion of the air: 1) $w = 0$; 2) 4.4 m/sec; 3) 3.8; 4) 8.35; 5) 11.6. t, ^{o}C.

In the general case, the local rate of change of the humidity and temperature inside the body is described by the known equations [1]

$$u' = \frac{\partial u}{\partial \tau} = \nabla\,[a_{\mathrm{m}}\,(\nabla u + \delta \nabla t_{\mathrm{s}})]\,, \tag{4}$$

$$t_{\mathrm{s}}' = \frac{\partial t_{\mathrm{s}}}{\partial \tau} = a \nabla^2 t_{\mathrm{s}} + \frac{r}{c_{\mathrm{s}}}\, \varepsilon u'\,. \tag{5}$$

With account for (3), instead of (4) we have the equation of mass transfer

$$u' = \frac{\partial u}{\partial \tau} = \nabla\,(a_{\mathrm{m}}^{\mathrm{eff}} \nabla u)\,. \tag{6}$$

Then by definition we have

$$\varphi' = \frac{\partial t_{\mathrm{s}}}{\partial u} = \frac{t_{\mathrm{s}}'}{u'}\,. \tag{7}$$

Here the relations $\partial x/\partial \tau = v_t = v_u$ are used.

Then, with account for Eq. (5),

$$\frac{dt_{\mathrm{s}}}{du} = \frac{a \nabla^2 u t_{\mathrm{s}} + \varepsilon r/c_{\mathrm{s}} u'}{u'} = \frac{a}{u'} \nabla^2 t_{\mathrm{s}} + \varepsilon \frac{r}{c_{\mathrm{s}}}\,, \tag{8}$$

$$\nabla t_{\mathrm{s}} = \varphi' \nabla u\,; \quad \nabla^2 t_{\mathrm{s}} = \varphi''\,(\nabla u)^2 + \varphi' \nabla^2 u\,, \tag{9}$$

where $\varphi'' = d^2 t_{\mathrm{s}}/du^2$.

The local rate of drying at any point according to (6) and (9) is

$$u' = \nabla\,(a_{\mathrm{m}}^{\mathrm{eff}} \nabla u) = \frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial u}\,(\nabla u)^2 + \frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}}\,\nabla t_{\mathrm{s}} \nabla u + a_{\mathrm{m}}^{\mathrm{eff}} \nabla^2 u = (\nabla u)^2 \left(\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial u} + \varphi'\,\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}}\right) + a_{\mathrm{m}}^{\mathrm{eff}} \nabla^2 u\,. \tag{10}$$

From the last equation we express $\nabla^2 u$ as

$$\nabla^2 u = \frac{u'}{a_{\mathrm{m}}^{\mathrm{eff}}} - \frac{(\nabla u)^2}{a_{\mathrm{m}}^{\mathrm{eff}}}\left(\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial u} + \varphi'\,\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}}\right). \tag{11}$$

The flow of moisture at any point of the body is

$$j = -\,\rho_{\mathrm{s}} a_{\mathrm{m}}^{\mathrm{eff}} \nabla u\,, \quad \nabla u = -\,j/(\rho_{\mathrm{s}} a_{\mathrm{m}}^{\mathrm{eff}})\,. \tag{12}$$

Substituting (11) and (12) into (9) and (8), after transformations we have

$$\frac{dt_{\mathrm{s}}}{du} = \frac{a}{u'}\,\frac{j^2}{\rho_{\mathrm{s}} a_{\mathrm{m}}^{\mathrm{eff}}}\left[\varphi'' - \frac{\varphi'}{a_{\mathrm{m}}^{\mathrm{eff}}}\left(\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial u} + \varphi'\,\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}}\right)\right] + \frac{a}{a_{\mathrm{m}}^{\mathrm{eff}}}\,\varphi' + \varepsilon\,\frac{r}{c_{\mathrm{s}}}\,. \tag{13}$$

In the case of invariance, the right-hand side of (13) must not depend on the local rate of drying u' and the density of the moisture flow j. The latter is possible if the expression in brackets is equal to zero.

Now about the invariance of the coefficient of evaporation. In the general case, we have

$$\varepsilon = \frac{u_1'}{u'} = \frac{\nabla\,(a_{\mathrm{m1}}^{\mathrm{eff}} \nabla u)}{u'} = \frac{1}{u'}\left\{\left[\frac{\partial a_{\mathrm{m1}}^{\mathrm{eff}}}{\partial u} + \varphi'\,\frac{\partial a_{\mathrm{m1}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}}\right](\nabla u)^2 + a_{\mathrm{m1}}^{\mathrm{eff}} \nabla^2 u\right\}. \tag{14}$$

Expressing ∇u and $\nabla^2 u$ in terms of u' and j and using (11) and (12), we have

$$\varepsilon = \frac{1}{u'}\,\frac{j^2}{\rho_{\mathrm{s}} a_{\mathrm{m}}^{\mathrm{eff}\,2}}\left[\frac{\partial a_{\mathrm{m1}}^{\mathrm{eff}}}{\partial u} + \varphi'\,\frac{\partial a_{\mathrm{m1}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}} - \frac{1}{a_{\mathrm{m}}^{\mathrm{eff}}}\left(\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial u} + \varphi'\,\frac{\partial a_{\mathrm{m}}^{\mathrm{eff}}}{\partial t_{\mathrm{s}}}\right)\right] + \frac{a_{\mathrm{m1}}^{\mathrm{eff}}}{a_{\mathrm{m}}^{\mathrm{eff}}}\,. \tag{15}$$

The quantity ε can be an invariant if the expression in brackets in (15) is equal to zero. Simultaneous fulfillment of this condition and the condition in (13) is possible only in the case where $a_{\mathrm{m}}^{\mathrm{eff}}$ = const, $a_{\mathrm{m1}}^{\mathrm{eff}}$ = const, and φ'' = 0. Then φ' = const, i.e., the dependence of t_{s} on u in the presence of invariance must be linear.

From (13) it follows that $a_{\mathrm{m}}^{\mathrm{eff}} = a$ at $\varepsilon = 0$. A small probability of this coincidence allows one to draw a conclusion: the invariant relation between temperature and humidity can exist only in the case of internal evaporation of moisture (when $\varepsilon \neq 0$) and at ε = const.

It is the constancy of the coefficients of transfer that forms the basis of the majority of the known solutions of the internal problem of heat and mass transfer; therefore it makes sense to analyze them. We consider, for example, one of the solutions of system (4)–(5) for a plate with the boundary conditions of the third kind [1, 3]:

$$t^* = \sum_{n=1}^{\infty} \sum_{i=1}^{2} C_{ni} \cos \nu_i\, \mu_n X \exp\,(-\,\mu_n^2\, \mathrm{Fo})\,,$$

$$u^* = \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{i=1}^{2} C_{ni}\;(1 - \nu_i^2) \cos \nu_i\, \mu_n X \exp\,(-\,\mu_n^2\, \mathrm{Fo})\,, \tag{16}$$

where

$$t^* = \frac{t_g - t_s}{t_g - t_{s0}}\, ; \quad u^* = \frac{u - u_{eq}}{u_0 - u_{eq}}\, ; \quad X = \frac{x}{R}\, .$$

Here, for convenience of analysis, the expressions for t^* and u^* are somewhat modified and the terms involving P^* are omitted, since, in moderate drying, the pressure inside the body does not change. The terms C_{ni}, μ_n, and ν_i are intricate complexes of internal and external parameters of drying that are determined by the corresponding formulas [3].

Eliminating Fo from (16), we can, in principle, obtain the dependence of the dimensionless temperature on the humidity that in the general case, is not an invariant and, moreover, is not linear. Thus, the constancy of the coefficients of transfer is not a sufficient condition of invariance. We try to consider the region of regular modes when only one first term remains significant in the entire infinite series (16). Then, we have

$$u^* \approx f_1\,(\mu X)\,\exp\,(-\,\mu^2 \mathrm{Fo})\, , \quad t^* \approx f_2\,(\mu X)\,\exp\,(-\,\mu^2 \mathrm{Fo})\, . \tag{17}$$

Here f_1 and f_2 are the functions known from (16). We can substantially simplify the form of Eq. (17) if we choose the moment of reckoning correctly. As the datum moment we take the moment when the body reaches a critical humidity u_{cr}, i.e., when the moisture content of the body surface decreases to a maximum hygroscopic value $u_{m.h}$. The indicated technique can be used if the material has both periods of drying. Then, at the end of the first period, the temperature at all points of the body is the same and equals t_w; moreover, of importance is the fact that in the first period a regular profile of moisture content is formed quickly, which is caused by the high values of a_m in capillary-porous materials. Thus, for Fo = Fo_{cr} we can write $t_s = t_w$, $u(X) = u_0(X)$ (the initial internal regular profile of moisture content), and

$$u^*_{cr} = \frac{u_0\,(X) - u_{eq}}{u_0 - u_{eq}} = f_1\,(\mu X)\,\exp\,(-\,\mu^2 \mathrm{Fo})\, , \quad t^*_{cr} = \frac{t_g - t_w}{t_g - t_{s0}} = f_2\,(\mu X)\,\exp\,(-\,\mu^2 \mathrm{Fo}_{cr})\, . \tag{18}$$

We divide the corresponding expressions in (17) by (18):

$$E\,(X) = \frac{u^*}{u^*_{cr}} = \frac{u_0 - u_{eq}}{u_0\,(X) - u_{eq}} = \exp\,(-\,\mu^2 \mathrm{Fo}^*)\, , \quad T\,(X) = \frac{t^*}{t^*_{cr}} = \frac{t_g - t_s}{t_g - t_w} = \exp\,(-\,\mu^2 \mathrm{Fo}^*)\, , \tag{19}$$

where $\mathrm{Fo}^* = \mathrm{Fo} - \mathrm{Fo}_{cr}$.

Integrating (19) over the entire volume of the body, we have for the volume-mean values

$$T = \frac{t_g - \bar{t}_s}{t_g - t_w} = \exp\,(-\,\mu^2 \mathrm{Fo}^*)\, , \quad E = \frac{\bar{u} - u_{eq}}{u_{cr} - u_{eq}} = \exp\,(-\,\mu^2 \mathrm{Fo}^*)\, . \tag{20}$$

Hence immediately follows the sought linear relation between temperature and humidity in the form

$$T = E\, . \tag{21}$$

Expression (21) can be treated as the first property of regular modes. Differentiating (20) with respect to time, we have

$$\frac{d\bar{u}}{d\tau} = -\,(u_{cr} - u_{eq})\,\exp\,(-\,\mu^2 \mathrm{Fo}^*)\,\mu^2 a_m / R^2$$

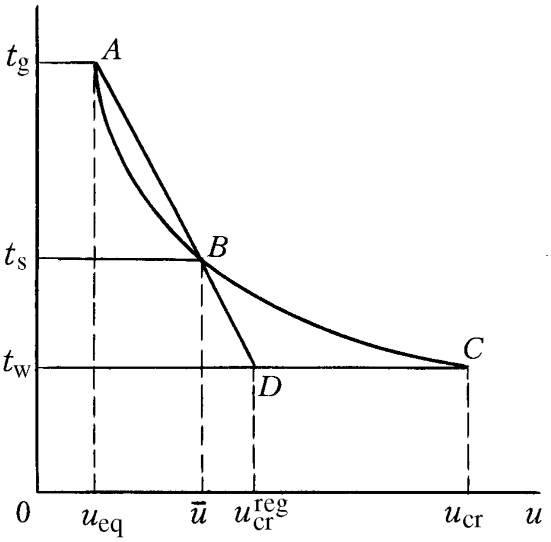


Fig. 2. Derivation of (27).

or with account for (20)

$$\frac{d\bar{u}}{d\tau} = -(u_{cr} - u_{eq})\, E\mu^2 a_m/R^2 . \tag{22}$$

When $E = 1$, the rate of drying is equal to N (the rate of drying in the first period); hence $\mu^2 = NR^2/(u_{cr} - u_{eq})/a_m$ and

$$-\frac{d\bar{u}}{d\tau} = NE , \quad -\frac{1}{N}\frac{d\bar{u}}{d\tau} = E . \tag{23}$$

This is the second property of regular modes in which the rate of drying decreases linearly with E.

We can formulate the third property of regular modes, comparing (23) and (21):

$$T = -\frac{1}{N}\frac{d\bar{u}}{d\tau}. \tag{24}$$

According to it, the dimensionless temperature of the body coincides with the dimensionless rate of drying.

In fact, the coefficients of transfer and external conditions of the process are not constant, as a result of which, strictly speaking, there must be no regular mode in the second period. As is known, an irregular mode takes place with a sharp jumpwise change in the process conditions. If changes occur very smoothly and slowly, for high values of a_m and a the internal fields of temperature and moisture content can rearrange themselves according to the changing conditions so that the mode of transfer differs very little from regular.

Therefore, we will consider that the properties of regular modes hold in the second period and under variable conditions. What changes must be made in expressions (21)–(24) in this case? Let, at a certain instant of time, the product have running parameters t_s and $\bar{u}$ that correspond to point B on the real temperature curve CBA (Fig. 2). The straight line DA reflects an ideal regular process with a constant quantity μ equal to its value at point B of the real curve. It is obvious that to each point of curve CBA there corresponds its own regular straight line. The points of intersection of the considered lines and the horizontal t_w = const determine the values of the critical moisture contents u_{cr}^{reg} and u_{cr}, which in the general case do not coincide.

Expression (21) holds for ideal straight lines of the DA type; therefore, strictly speaking, it must involve u_{cr}^{reg} rather than u_{cr}, i.e.,

$$T = E_{\text{reg}} \, , \quad E_{\text{reg}} = (\bar{u} - u_{\text{eq}})/(u_{\text{cr}}^{\text{reg}} - u_{\text{eq}}) \, . \tag{25}$$

The quantity $u_{\text{cr}}^{\text{reg}}$ in Fig. 2 depends on the position of point B and the shape of the considered curve, i.e.,

$$u_{\text{cr}}^{\text{reg}} - u_{\text{eq}} = f_0 \, (\bar{u} - u_{\text{eq}}) \, . \tag{26}$$

If there is a set of curves recorded in different modes rather than one curve, then dependence (26) can be generalized to the entire set. Then we have a dependence that is unique for all curves that characterizes the product studied: $u_{\text{cr}}^{\text{reg}} - u_{\text{eq}} = f(u - u_{\text{eq}}, t_{\text{s}} - t_{\text{w}}, t_{\text{g}} - t_{\text{w}})$, or in dimensionless form

$$E_{\text{reg}} = \varphi \, (E, T) \, . \tag{27}$$

On the other hand, if at any point of the real curve the mode of heat and mass transfer differs little from regular, equality (25) must hold. Then, comparing (25) and (27), we have

$$T = \varphi \, (E, T) \, . \tag{28}$$

Equation (28) expresses the single-valued dependence of T on E that can be written in explicit form:

$$T = F \, (E) \, . \tag{29}$$

Thus, in the general case, the dependence of T on E, in contrast to (21), is nonlinear; moreover, by virtue of the single-valuedness of the function F it does not depend on the mode of drying, i.e., it is an invariant for the considered material which satisfies the regularity conditions. The form of the function F can easily be found from experiments. Knowing F, one can find the temperature of the material at any point of the drying chamber if at this point $\bar{u}$, t_{g}, t_{w}, u_{cr}, and u_{eq} are known. Consequently, it is not necessary to solve the equation of heat transfer (5).

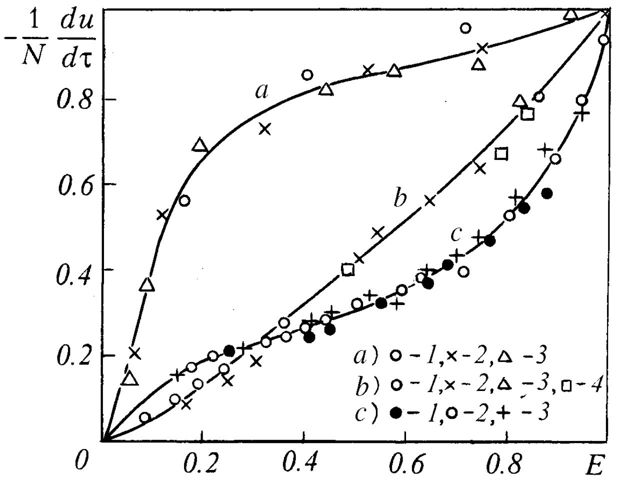
Equation (29) expresses the property of similarity of heat and mass transfer processes in drying: the curves of the dependence of the temperature of the material on its humidity in different modes are similar and converge to one curve when dimensionless variables are used. We note that the similarity is not identical to the invariance in the sense of a UFD, but for thin specimens we can neglect slight changes in the critical humidity u_{cr} in different hydrodynamic modes; then (29) will express an approximate dependence of t_{s} on $\bar{u}$. The latter is observed in Fig. 1.

We now refer to (23). Substituting the value of E_{reg} from (27) instead of E into it, we obtain with account for (29):

$$-\frac{1}{N} \frac{d\bar{u}}{d\tau} = \Phi \, (E) \, . \tag{30}$$

Here, the similarity of the processes manifests itself: the dimensionless rate of drying is a single-valued function of the dimensionless humidity E. The similarity property makes it possible to represent the equation of kinetics in the form of the product of two independent cofactors: the rate of drying in the first period N that allows for the mode (including hydrodynamic) and the invariant $\Phi(E)$. The form of the latter is found by differentiation of the experimental curve of kinetics. It is, however, of importance that Eq. (30) be obtained by processing a series of experiments indicating the presence of similarity. Kinetic similarity is confirmed by numerous experimental data (Fig. 3, data of [1–4]).

We note that expressions (29) and (30) are two sides of one phenomenon – similarity – and they are inseparable, i.e., if the temperature similarity is observed, then there must obligatorily exist the kinetic simi-



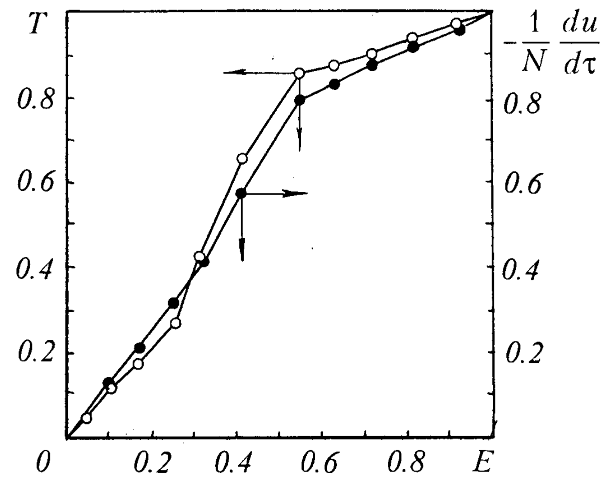


Fig. 3. Similarity of kinetic curves for some materials: a) cloth: 1) τ = 80°C; 2) 60; 3) 40; b) clay: 1) 45; 2) 35; 3) 25; 4) 15; c) bread: 1) 125; 2) 100; 3) 80.

Fig. 4. Comparison of dimensionless temperature and kinetic curves of drying of a condensed milk droplet.

larity, and vice versa. The relation between them yields the third property of regular modes. Comparing (24) with (29) and (30), we find

$$F(E) = \Phi(E)\,, \tag{31}$$

i.e., the temperature and kinetic invariants for one and the same product must be equal. Then, using the known temperature curve, we can find the unknown curve of the rate of drying and vice versa. The property indicated is also confirmed experimentally (Fig. 4, data of [4]). The small differences in the curves can be explained by the different accuracy of determination of the temperature and the rate of drying, since the first is measured directly and the second using the method of graphic differentiation.

When N = const, we can separate the variables in (30); after integration we have

$$\int_1^E \frac{d\bar{u}}{\Phi(E)} = -N\tau \quad \text{or} \quad E = F_1\left(\frac{N\tau}{u_{\text{cr}} - u_{\text{eq}}}\right). \tag{32}$$

According to (32), all experimental data obtained in different modes are integrated into one self-similar curve constructed in the coordinates $E - N\tau/(u_{\text{cr}} - u_{\text{eq}})$; the origin and the onset of the second period must coincide. In this case, the function F_1 is also an invariant; however, its practical value is bounded by the modes with N = const.

The similarity of processes offers a researcher a serious instrument for calculation and analysis in the form of invariants (29) and (30), which can be considered to be approximate integrals of system (4)–(5). The advantage of these invariants over any analytical solutions lies in the automatic account for all nonlinearities, including shrinkage, change in the internal structure of a body, etc. Here, it also becomes unnecessary to experimentally determine variable coefficients of the system.

We should mention one more remarkable property of similarity: according to (29) and (30), the running temperature of the body and rate of drying do not depend on the prehistory of the process, i.e., they are determined only by the running parameters of the process at a given instant of time. The indicated property allows one to use similarity for calculation of processes with any smooth changes in external conditions.

NOTATION

a and a_m, coefficients of thermal diffusivity and mass transfer, m^2/sec; c, heat capacity; j, density of the moisture flow; N, rate of drying in the first period, 1/sec; r, heat of moisture evaporation; R, radius of the body; t, temperature; T, dimensionless temperature; u, moisture content, kg/kg; v, velocity of displacement of the field; w, velocity of motion of the air; x, coordinate; δ, relative coefficient of thermal diffusion (dimensionless); ε, coefficient of evaporation (dimensionless); ρ, density; τ, time; $\mathrm{Fo} = a_m\tau/R^2$, Fourier number; $\mathrm{Ko} = r(u_0 - u_{eq})/[c_s(t_g - t_{s0})]$, Kossovich number. Subscripts and superscripts: g, gas; 0, at the inlet; eq, equilibrium; cr, critical, w, wet thermometer; m.h, maximum hygroscopic; reg, regular; s, solid phase; eff, effective.

REFERENCES

1. A. V. Luikov, in: *Theory of Drying* [in Russian], Moscow (1968), pp. 126, 163, and 233.
2. M. M. Razin, *Inzh.-Fiz. Zh.*, **63**, No. 4, 442–448 (1992).
3. A. V. Luikov and Yu. A. Mikhailov, in: *Theory of Heat and Mass Transfer* [in Russian], Moscow (1963), p. 230.
4. A. S. Ginzburg, in: *Principles of the Theory and Technology of Drying of Foodstuffs* [in Russian], Moscow (1973), pp. 297, 381, and 257.